Lecture 31

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1 Symmetric matrices

As we saw before, sometimes it happens that the matrix is not diagonalizable over $\mathbb{R}$.. It may happen, for example, when some roots of the characteristic polynomial are complex. In this lecture we will consider the properties of symmetric matrices.

Theorem 1.1. *Let A be a real symmetric matrix. Then each root λ of its characteristic polynomial is real.*

Moreover, for symmetric matrices, eigenvectors, corresponding to different eigenvalues are not only linearly independent, but even orthogonal:

Theorem 1.2. *Let A be a real symmetric matrix. Let λ_1 and λ_2 be different eigenvalues of A, and e_1 and e_2 are corresponding eigenvectors. Then e_1 and e_2 are orthogonal, i.e. $\langle e_1, e_2 \rangle = 0$.*

So, these 2 theorems give us the following main result about symmetric matrices:

Theorem 1.3. *Let A be a real symmetric matrix. Then there exists a matrix C such that $D = C^{-1}AC$ is diagonal. Moreover, columns of a matrix C are orthogonal.*

We can find a matrix C such that its columns are not simply orthogonal to each other, but have norm equal to 1. To do this we should take normalizations of found eigenvectors, and write them as columns of a matrix C. Such matrix C satisfy the property $CC^\top = I$, or, which is equivalent, $C^{-1} = C^\top$. Such matrices are called **orthogonal matrices**.

We'll demonstrate it in the following example.

Example 1.4. *Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then the characteristic polynomial is $p_A(\lambda) = \lambda^2 - 4\lambda + 3$. The roots of it are real, as expected: $\lambda_1 = 1$ and $\lambda_2 = 3$. So, the diagonal form of this matrix is $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.*

Now let's find eigenvectors.

$\boldsymbol{\lambda_1 = 1}$. *Subtracting* $\lambda_1 = 1$ *from diagonal elements we get* $A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *so the equation for eigenvectors is* $x + y = 0$, *so, one of the eigenvectors is* $e_1 = (1, -1)$. *Norm of it is* $\sqrt{1+1} = \sqrt{2}$, *thus normalization of it is* $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

$\boldsymbol{\lambda_2 = 3}$. *Subtracting* $\lambda_2 = 3$ *from diagonal elements we get* $A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, *so the equation for eigenvectors is* $-x + y = 0$, *so, one of the eigenvectors is* $e_2 = (1, 1)$. *Norm of it is* $\sqrt{1+1} = \sqrt{2}$, *thus normalization of it is* $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

It can be easily seen that these eigenvectors are orthogonal to each other. So, matrix C *is* $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. *Now,* $C^{-1} = C^{\top} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

Now, one can easily check that $D = C^{-1}AC$.

The proofs of the theorems stated above are difficult, and require additional technics, not considered in this course, so we will post them in the addendum to this lecture.